A discrete magnetostatic solver in Matlab/Octave

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1 Introduction

In the open-rTMS project a complex magnetic field is applied to the human brain. The whole setup consists of several coils placed around the skull. I assume, that the properties of the brain are LTI (linear, time invariant), so that a simple superposition of the fields from the different coils can be calculated the get the field distribution inside the brain. The coils I assume to be small compared to the distance between them and the field strength is small as well, so that the influence of the coils towards each other can also be neglected.

These assumptions should work out well, if following types of coils are used:

- a simple core made from a screw with some turns of copper. A ferrite core is not needed, because of the low frequency.
- flat coils without core. In this case also overlapping coils are not a problem.

2 Theory

To calculate the magnetic fieldlines we first calculate the vector potential A. After A is computed we can calculate the magnetic field $B = \mathrm{rot}(A) = \nabla \times A$.
To calculate A we start with the Poisson equation in 3D (μ: permeability of material, j: current density in coil):

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\cdot\frac{\partial A}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\mu}\cdot\frac{\partial A}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\mu}\cdot\frac{\partial A}{\partial z}\right)=-j \tag{1}$$

To simplify the solution of equation (1) the problem is simplified to a 2D rotational system (Figure 2).

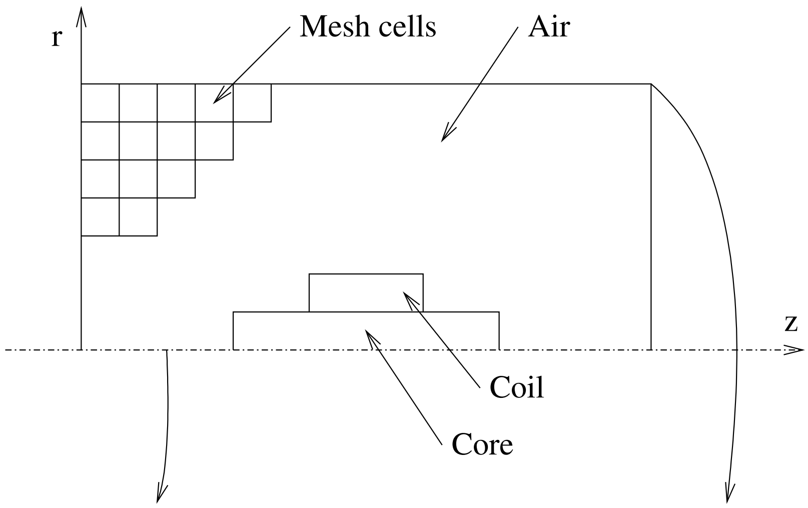

Figure 1: Rotational symmetric setup

in this rotational case the Poisson equation (1) becomes:

$$\frac{\partial}{\partial r}\left(\frac{1}{r\cdot\mu}\cdot\frac{\partial(r\cdot A)}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\mu}\cdot\frac{\partial A}{\partial z}\right)=-j \tag{2}$$

This equation is solved for a grid of cells. The terms in (2) are rewritten as finite differences: The inner partial differentials according to figure 2 will be

$$\frac{\partial(r\cdot A)}{\partial r}\Rightarrow \frac{(r+\Delta r/2)\cdot A(r+\Delta r/2)-(r-\Delta r/2)\cdot A(r-\Delta r/2)}{\Delta r} \tag{3}$$

and

$$\frac{\partial A}{\partial z}\Rightarrow \frac{A(z+\Delta z/2)-A(z-\Delta z/2)}{\Delta z} \tag{4}$$

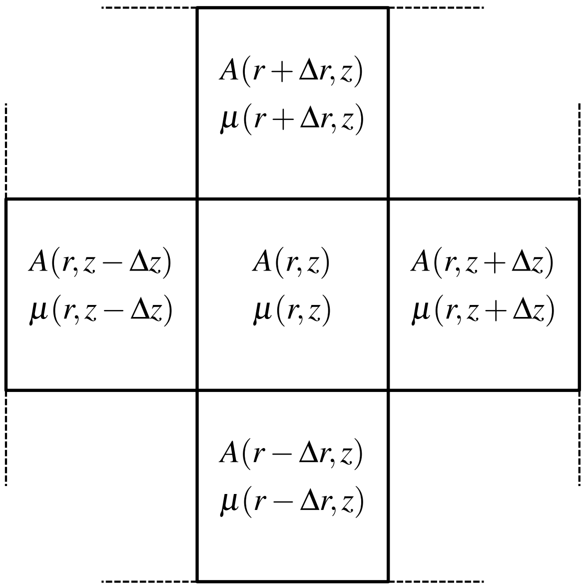

Figure 2: Cells in the grid

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For the whole expression (2) we get

$$\frac{\partial}{\partial r}\left(\frac{1}{r\cdot\mu}\cdot\frac{\partial(r\cdot A)}{\partial r}\right)\Rightarrow$$
$$\frac{(r+\Delta r)\cdot A(r+\Delta r)-r\cdot A}{(r+\Delta r/2)\cdot\mu(r+\Delta r/2)\cdot\Delta r^2}-\frac{r\cdot A-(r-\Delta r)\cdot A(r-\Delta r)}{(r-\Delta r/2)\cdot\mu(r-\Delta r/2)\cdot\Delta r^2} \tag{5}$$

and

$$\frac{\partial}{\partial z}\left(\frac{1}{\mu}\cdot\frac{\partial A}{\partial z}\right)\Rightarrow$$
$$\frac{\frac{A(z+\Delta z)-A}{\mu(z+\Delta z/2)\cdot\Delta z}-\frac{A-A(z-\Delta z)}{\mu(z-\Delta z/2)\cdot\Delta z}}{\Delta z}=$$
$$\frac{A(z+\Delta z)-A}{\mu(z+\Delta z/2)\cdot\Delta z^2}-\frac{A-A(z-\Delta z)}{\mu(z-\Delta z/2)\cdot\Delta z^2} \tag{6}$$

.

As it can be seen from figure 2 μ is used in equations (5) and (6) on a 45° rotated grid between the cells.

For this grid is hard to define in Matlab/Octave without making errors, we define μ at the very same points as A (as shown in figure 2) and use terms like

$$\mu(z+\Delta z/2)=\frac{\mu(z)+\mu(z+\Delta z)}{2} \tag{7}$$

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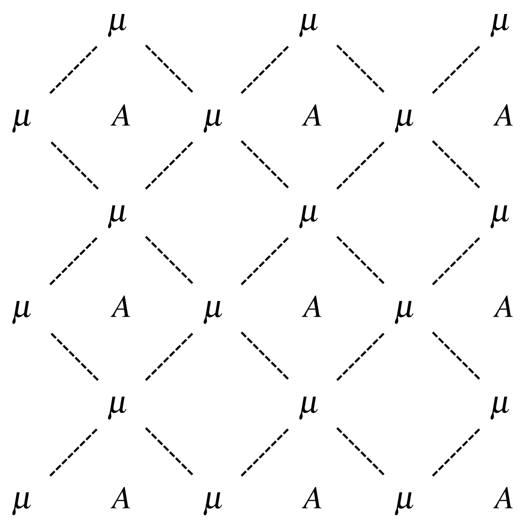

Figure 3: μ inbetween A

Equations (5) and (6) become

$$\frac{2\cdot(r+\Delta r)\cdot A(r+\Delta r)-2\cdot r\cdot A}{(r+\Delta r/2)\cdot(\mu+\mu(r+\Delta r))\cdot\Delta r^2}+\frac{2\cdot(r-\Delta r)\cdot A(r-\Delta r)-2\cdot r\cdot A}{(r-\Delta r/2)\cdot(\mu+\mu(r-\Delta r))\cdot\Delta r^2} \tag{8}$$

and

$$\frac{2\cdot A(z+\Delta z)-2\cdot A}{(\mu+\mu(z+\Delta z))\cdot\Delta z^2}+\frac{2\cdot A(z-\Delta z)-2\cdot A}{(\mu+\mu(z-\Delta z))\cdot\Delta z^2} \tag{9}$$

.

Everything put together and all the different A taken apart:

$$\begin{aligned}
&A(r+\Delta r,z)\cdot\frac{2\cdot(r+\Delta r)}{(r+\Delta r/2)\cdot(\mu+\mu(r+\Delta r))\cdot\Delta r^2}\\
&+A(r-\Delta r,z)\cdot\frac{2\cdot(r-\Delta r)}{(r-\Delta r/2)\cdot(\mu+\mu(r-\Delta r))\cdot\Delta r^2}\\
&+A(r,z+\Delta z)\cdot\frac{2}{(\mu+\mu(z+\Delta z))\cdot\Delta z^2}\\
&+A(r,z-\Delta z)\cdot\frac{2}{(\mu+\mu(z-\Delta z))\cdot\Delta z^2}\\
&-A(r,z)\cdot\left(\frac{2\cdot r}{(r+\Delta r/2)\cdot(\mu+\mu(r+\Delta r))\cdot\Delta r^2}\right.\\
&\quad+\frac{2\cdot r}{(r-\Delta r/2)\cdot(\mu+\mu(r-\Delta r))\cdot\Delta r^2}\\
&\quad\left.+\frac{2}{(\mu+\mu(z+\Delta z))\cdot\Delta z^2}+\frac{2}{(\mu+\mu(z-\Delta z))\cdot\Delta z^2}\right)\\
&=-j
\end{aligned} \tag{10}$$

This lengthy equation (10) is solved for $A(r,z)$.

2.1 Boundary conditions

In theory the magnetic field lines (or actually field planes) extent into infinity. The simulated volume is finite.

3 Solution

3.1 Iterative solution

A μ- and a j-matrix are given. The A-matrix is initialized to zero.
The matrix A is calculated iteratively. For every step $k = 1, 2, 3, ...$

- Calculate $A^k(r,z)$ from equation (10) using the values of the surrounding cells from the previous step ($A^{k-1}(r+\Delta r,z)$, $A^{k-1}(r-\Delta r,z)$, $A^{k-1}(r,z+\Delta z)$, $A^{k-1}(r,z-\Delta z)$).
- Calculate the error in cell $A(r,z)$: $err = A^k(r,z) - A^{k-1}(r,z)$.
- Correct the error in cell $A(r,z)$: $A^k(r,z) = A^{k-1}(r,z) + err \cdot V$
- Repeat.

V is a twiddel-factor:

$$\begin{aligned} V &< 0 \text{ instable} \\ V &\to 0 \text{ exact solution} \\ V &\to 1 \text{ fast solution} \\ 1 < V &< 2 \text{ oscillates but might become stable eventually} \\ V &> 2 \text{ instable} \end{aligned}$$

3.1.1 Hint

If one looks closely at the error calculation and correction one can see a discrete 1st order lowpass filter (PT-1) here.
Transfer behavior from x^k to y^k:

$$\begin{aligned} y^k &= y^{k-1} + V \cdot (x^k - y^{k-1}) \\ Y(z) &= Y(z) \cdot z^{-1} + V \cdot (X(z) - Y(z) \cdot z^{-1} \\ \frac{Y(z)}{X(z)} &= \frac{V}{1 + (V-1) \cdot z^{-1}} \end{aligned}$$

Transfer-function:

$$G(z) = \frac{V \cdot z}{z + (V-1)}$$

The filter has a zero at $0 + \imath 0$ and a pole at $(1-V) + \imath 0$. For $V < 0$ or $V > 2$ the pole leaves the unit circle and the filter becomes instable.

3.2 FEM Solution

The equation 1 can be rewritten in the form $C \cdot A = j$ with A a vector of the vectorpotentials in very cell, j a vector of the current density in every cell and C a matrix of constants. With a simple inversion of C we get the solution for A:

$$A = C^{-1} \cdot j$$

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4 Matlab/Octave code